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ON LEBESGUE'S CONSTANTS IN THE THEORY OF FOURIER'S SERIES.*

By T. H. Gronwall.

The definition of the nth Lebesgue constant ρ_n is

$$\rho_n = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\left|\frac{\sin(2n+1)t}{\sin t}\right|dt,$$

and by certain transformations of this integral, I have shown in a previous paper† that

$$\rho_{n+1} > \rho_n, \qquad (n = 1, 2, 3, \cdots) \tag{1}$$

In the latter part of the paper referred to, I used Fejér's formula

$$\rho_n = \frac{2}{\pi}\sum_{\mu=1}^{n}\frac{1}{\mu}\tan\frac{\mu\pi}{2n+1} + \frac{1}{2n+1} \tag{2}$$

to obtain a general asymptotic expression for ρ_n; and from the point of view of unity in method, it is desirable also to deduce (1) directly from (2). This is the purpose of the present note.

From (2) we obtain at once

$$\begin{aligned}\rho_{n+1} - \rho_n &= \frac{2}{\pi}\cdot\frac{1}{n+1}\tan\frac{n+1}{2n+3}\pi + \frac{2}{\pi}\sum_{\mu=1}^{n}\frac{1}{\mu}\left(\tan\frac{\mu\pi}{2n+3} - \tan\frac{\mu\pi}{2n+1}\right)\\ &\qquad + \frac{1}{2n+3} - \frac{1}{2n+1}\\ &= \frac{2}{\pi}\cdot\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)} - \frac{2}{\pi}\sum_{\mu=1}^{n}\frac{1}{\mu}\cdot\frac{\sin\dfrac{2\mu\pi}{(2n+1)(2n+3)}}{\cos\dfrac{\mu\pi}{2n+1}\cos\dfrac{\mu\pi}{2n+3}}\\ &\qquad - \frac{2}{(2n+1)(2n+3)}\\ &> \frac{2}{\pi}\cdot\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)}\\ &\qquad - \frac{4}{(2n+1)(2n+3)}\sum_{\mu=1}^{n}\frac{1}{\cos\dfrac{\mu\pi}{2n+1}\cos\dfrac{\mu\pi}{2n+3}}\\ &\qquad - \frac{2}{(2n+1)(2n+3)},\end{aligned}$$

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[*] Presented to the American Mathematical Society, February 22, 1913.

[†] "Über die Lebesgueschen Konstanten bei den Fourierschen Reihen," Math. Annalen, vol. 72 (1912), pp. 244–261.

or introducing the summation index $\nu = n - \mu$,

$$\rho_{n+1} - \rho_n > \frac{2}{\pi}\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)}$$

$$- \frac{4}{(2n+1)(2n+3)}\sum_{\nu=0}^{n-1}\frac{1}{\sin\frac{2\nu+1}{2n+1}\frac{\pi}{2}\sin\frac{2\nu+3}{2n+3}\frac{\pi}{2}} \tag{3}$$

$$- \frac{2}{(2n+1)(2n+3)}.$$

From the formula

$$\frac{1}{\sin x} = \cot\frac{x}{2} - \cot x$$

and the development of the cotangent in partial fractions it is seen that

$$\frac{1}{x}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \sum_{n=1}^{\infty}\left(\frac{2}{n^2\pi^2 - x^2} - \frac{1}{n^2\pi^2 - x^2/4}\right),$$

and since each term in the series increases monotonely with x for $0 < x < \pi/2$, the same is true for the left hand expression, so that

$$\frac{1}{x}\left(\frac{1}{\sin x} - \frac{1}{x}\right) < \frac{2}{\pi}\left(1 - \frac{2}{\pi}\right) \qquad \left(0 < x < \frac{\pi}{2}\right),$$

or denoting the right hand constant by a,

$$\frac{1}{\sin x} < \frac{1}{x} + ax \qquad \left(0 < x < \frac{\pi}{2}\right). \tag{4}$$

In the same way, it is seen that

$$\frac{1}{x}\left(\frac{1}{x} - \cot x\right) = \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2 - x^2}$$

increases monotonely with x for $0 < x < \pi/2$, so that

$$\cot x > \frac{1}{x} - \frac{4}{\pi^2}x \qquad \left(0 < x < \frac{\pi}{2}\right).$$

We have, therefore,

$$\begin{aligned} \frac{2}{\pi}\cdot\frac{1}{n+1}\cot\frac{\pi}{2(2n+3)} &> \frac{4}{\pi^2}\cdot\frac{2n+3}{n+1} - \frac{4}{\pi^2}\cdot\frac{1}{(n+1)(2n+3)} \\ &= \frac{4}{\pi^2}\cdot\frac{4n+8}{2n+3} = \frac{8}{\pi^2} + \frac{8}{\pi^2}\cdot\frac{1}{2n+1} - \frac{16}{\pi^2}\cdot\frac{1}{(2n+1)(2n+3)}. \end{aligned} \tag{5}$$

Now (4) gives

$$\frac{4}{(2n+1)(2n+3)}\cdot\frac{1}{\sin\frac{2\nu+1}{2n+1}\frac{\pi}{2}\sin\frac{2\nu+3}{2n+3}\frac{\pi}{2}}<\frac{16}{\pi^2}\frac{1}{(2\nu+1)(2\nu+3)}$$

$$+4a\left(\frac{1}{(2n+1)^2}\cdot\frac{2\nu+1}{2\nu+3}+\frac{1}{(2n+3)^2}\cdot\frac{2\nu+3}{2\nu+1}\right)+a^2\pi^2\frac{(2\nu+1)(2\nu+3)}{(2n+1)^2(2n+3)^2}.$$

We have

$$\sum_{\nu=0}^{n-1}\frac{16}{\pi^2}\cdot\frac{1}{(2\nu+1)(2\nu+3)}=\frac{8}{\pi^2}\sum_{\nu=0}^{n-1}\left(\frac{1}{2\nu+1}-\frac{1}{2\nu+3}\right)=\frac{8}{\pi^2}\left(1-\frac{1}{2n+1}\right)$$

and

$$\frac{1}{(2n+1)^2}\cdot\frac{2\nu+1}{2\nu+3}+\frac{1}{(2n+3)^2}\cdot\frac{2\nu+3}{2\nu+1}=\frac{1}{(2n+1)^2}-\frac{2}{(2n+1)^2}\cdot\frac{1}{2\nu+3}$$

$$+\frac{1}{(2n+3)^2}+\frac{2}{(2n+3)^2}\cdot\frac{1}{2\nu+1}<\frac{1}{(2n+1)^2}+\frac{1}{(2n+3)^2}$$

$$+\frac{2}{(2n+3)^2}\left(\frac{1}{2\nu+1}-\frac{1}{2\nu+3}\right),$$

whence

$$\sum_{\nu=0}^{n-1}\left(\frac{1}{(2n+1)^2}\cdot\frac{2\nu+1}{2\nu+3}+\frac{1}{(2n+3)^2}\cdot\frac{2\nu+3}{2\nu+1}\right)<\frac{n}{(2n+1)^2}+\frac{n}{(2n+3)^2}$$

$$+\frac{2}{(2n+3)^2}\left(1-\frac{1}{2n+1}\right)=\frac{1}{2n+1}-\frac{1}{(2n+1)(2n+3)}$$

$$-\frac{8n+6}{(2n+1)^2(2n+3)^2}<\frac{1}{2n+1}-\frac{1}{(2n+1)(2n+3)}.$$

Furthermore we have

$$\frac{1}{(2n+1)^2(2n+3)^2}\sum_{\nu=0}^{n-1}(2\nu+1)(2\nu+3)$$

$$=\frac{1}{(2n+1)^2(2n+3)^2}\sum_{\nu=0}^{n-1}\frac{(2\nu+1)(2\nu+3)(2\nu+5)-(2\nu-1)(2\nu+1)(2\nu+3)}{6}$$

$$=\frac{1}{(2n+1)^2(2n+3)^2}\cdot\frac{(2n-1)(2n+1)(2n+3)+3}{6}$$

$$=\frac{1}{6}\left(\frac{1}{2n+1}-\frac{4}{(2n+1)(2n+3)}+\frac{3}{(2n+1)^2(2n+3)^2}\right)$$

$$\leqq\frac{1}{6}\cdot\frac{1}{2n+1}-\frac{1}{2}\cdot\frac{1}{(2n+1)(2n+3)},$$

so that we finally obtain

$$\frac{4}{(2n+1)(2n+3)}\sum_{\nu=0}^{n-1}\frac{1}{\sin\frac{2\nu+1}{2n+1}\frac{\pi}{2}\sin\frac{2\nu+3}{2n+3}\frac{\pi}{2}} < \frac{8}{\pi^2}\left(1-\frac{1}{2n+1}\right) \tag{6}$$

$$+\frac{4a}{2n+1}-\frac{4a}{(2n+1)(2n+3)}+\frac{a^2\pi^2}{6}\cdot\frac{1}{2n+1}-\frac{a^2\pi^2}{2}\frac{1}{(2n+1)(2n+3)}.$$

From (3), (5) and (6) it is now seen that

$$\rho_{n+1}-\rho_n > \frac{1}{2n+1}\left[\left(\frac{16}{\pi^2}-4a-\frac{a^2\pi^2}{6}\right)-\left(\frac{16}{\pi^2}+2-4a-\frac{a^2\pi^2}{2}\right)\frac{1}{2n+3}\right],$$

or introducing

$$a=\frac{2}{\pi}\left(1-\frac{2}{\pi}\right)$$

and calculating the numerical values of the constants

$$\rho_{n+1}-\rho_n > \frac{1}{2n+1}\left(0.607-\frac{2.432}{2n+3}\right). \tag{7}$$

This expression being evidently positive for $n \geqq 1$, our formula (1) is proved.

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